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Explicit Generalization of Lagrange's Equations for Hybrid Coordinate Dynamical Systems

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An explicit generalization of the classical Lagrange's equations (for discrete coordinate dynamical systems) to cover a large family of multibody hybrid discrete/distributed parameter systems is presented. The coupled system of ordinary and partial differential equations follows directly from spatial and time differentiation of various Lagrangian functionals, whereas the boundary conditions are directly established from another explicit set of symbolic variational equations. Five illustrative examples are presented.

Introduction

WE consider a family of multibody hybrid discrete/distributed parameter systems that can be regarded as consisting of a collection of interconnected rigid and elastic bodies. Such models are useful for dynamics and control analysis of flexible spacecraft. The equations of motion are hybrid, in the sense that the rigid-body motions are described by discrete time-varying coordinates, and the elastic motions are described by time- and space-varying coordinates; the resulting hybrid system of ordinary and partial integro-differential equations embodies significant coupling between the rigid-body and elastic motions.[1]

Meirovitch[2] extended the classical Lagrange's equations for hybrid systems using the extended Hamilton's principle. Although Meirovitch found the correct forms for the hybrid system, his equations embodied a differential operator that must be developed through integration by parts for each specific application. Also, the boundary condition operator in Meirovitch's developments must be found by integration by parts for each specific application. Berbyuk and Demidyuk[3] formulated the dynamic equations and boundary conditions for a specific mechanical system (two-link manipulator with one rigid link and one flexible link) by means of the extended Hamilton's principle. In deriving the kinetic energy and boundary conditions, they included the effects of end payload. Low and Vidyasagar[4] presented a procedure for deriving dynamic equations for manipulators containing both rigid and flexible links. They proposed a method for producing a compact symbolic expression for the equation of flexible manipulator systems. As they mentioned, their boundary conditions do not make allowance for ends that involve discrete elements, such as lumped masses and springs. Of course, Hamilton's principle can produce the appropriate boundary conditions in such cases, but the procedure is system-specific and tedious, especially when dealing with multiple-connected flexible bodies.

We were motivated by Meirovitch's developments to establish, at least for significant classes of systems, explicit Lagrange differential equations and boundary conditions that make allowance for lumped masses, springs, and similar forces at the boundaries. In essence, we seek to symbolically carry out the integration by parts once and for all for a large class of systems. In the present paper, explicit Lagrange's equations and the associated boundary conditions are developed for hybrid systems that have one spatial variable per elastic body. These developments are a portion of a recent study documented in greater detail in Ref. 1. The explicit governing equations and the associated boundary conditions are identical to those derivable from the extended Hamilton's principle,[2,5] but do not require the tedious system-specific variational arguments and integration by parts that are generally associated with the application of the extended Hamilton's principle to multiple flexible body systems.[3] As illustrations, the specific governing equations and boundary conditions for five example hybrid systems are obtained from the resulting general equations of this paper.

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Explicit Form of the Governing Equations and the Boundary Conditions for Hybrid Systems

We consider a hybrid coordinate dynamical system and assume that the Lagrangian $L = T - V$, in which T is the kinetic energy and V is the potential energy, can be written in the general form $L = L(t, P, q_i, \dot{q}_i, w_j, \dot{w}_j, w_j', w_j'')$, where $q_i = q_i(t)$ $(i = 1,2,\ldots,m)$ are generalized coordinates describing rigid-body motions of the hybrid system and $w_j = w_j(P,t)$ $(j = 1,2,\ldots,n)$ are distributed coordinates describing elastic motions relative to the rigid-body motions of an undeformed body-fixed spatial position P. We define $\boldsymbol{q}$ and $\boldsymbol{w}$ generalized coordinate vectors such as $\boldsymbol{q} = [q_1, q_2, \ldots, q_m]^T$ and $\boldsymbol{w} = [w_1, w_2, \ldots, w_n]^T$. Overdots designate derivatives with respect to time, and primes designate derivatives with respect to the spatial position.

First we consider the case that there is only one elastic domain. For convenience, we assume that the Lagrangian consists of three terms such as $L = L_D + \int_D \hat{L}\, \mathrm{d}D + L_B$, where D is the domain of the undeformed flexible body; L_D is the discrete portion of L and is a function of t, $\boldsymbol{q}$, and $\dot{\boldsymbol{q}}$ in the Lagrangian; $\hat{L}$ is the Lagrangian density that is a function of $\boldsymbol{q}, \dot{\boldsymbol{q}}, \dot{\boldsymbol{w}}, \boldsymbol{w}, \boldsymbol{w}', \boldsymbol{w}''$, and (P,t); L_B, which is a function of $\boldsymbol{w}(l)$, $\dot{\boldsymbol{w}}(l)$, $\boldsymbol{w}'(l)$, $\dot{\boldsymbol{w}}'(l)$, $\boldsymbol{q}$, and $\dot{\boldsymbol{q}}$, is the "boundary term" portion of the Lagrangian energy functional, which in turn depends on the boundary motions.

Next, we consider the nonconservative virtual work of hybrid systems. The virtual work can be written in the form $\delta W_{nc} = \boldsymbol{Q}^T \delta\boldsymbol{q} + \int_D \hat{\boldsymbol{f}}^T \delta\boldsymbol{w}\, \mathrm{d}D + \boldsymbol{f}_1^T \delta\boldsymbol{w}(l) + \boldsymbol{f}_2^T \delta\boldsymbol{w}'(l)$. Here, $\boldsymbol{Q}$ is the nonconservative generalized force vector associated with $\boldsymbol{q}$, $\hat{\boldsymbol{f}}$ is the nonconservative generalized force density vector associated with $\boldsymbol{w}$, $\delta\boldsymbol{q}$ and $\delta\boldsymbol{w}$ are associated virtual displacements, and $\boldsymbol{f}_1^T \delta\boldsymbol{w}(l)$ and $\boldsymbol{f}_2^T \delta\boldsymbol{w}'(l)$ are the nonconservative virtual work that depends on the boundary forces and associated boundary virtual displacements. If $\boldsymbol{w}$ is a linear displacement vector, then $\boldsymbol{f}_1$ is the nonconservative boundary force vector associated with $\delta\boldsymbol{w}(l)$ and $\boldsymbol{f}_2$ is the nonconservative boundary torque vector associated with $\delta\boldsymbol{w}'(l)$.

Now we consider the simplest case of one spatial variable and derive an explicit version of Lagrange's equations for this class of hybrid systems. Usually, one spatial variable is used to identify the undeformed flexible body position in one domain. For certain symmetric configurations (for example, see Ref. 6), one spatial variable can represent multiple domains if symmetric or antisymmetric deformation is assumed. We assume that the kinetic energy, potential energy, and Lagrangian are expressed by Eqs. (1), (2), and (3), respectively, and $q = q(t)$, $w = w(x,t)$, and $w(l)$ represents $w(x,t)|_l$. Also, we suppress the appearance of t everywhere for notational compaction.

$$T = T_D(q,\dot q) + \int_{l_0}^{l} \hat T(q,\dot q,\dot w,w,w',w'')\,\mathrm{d}x + T_B\big[w(l),\dot w(l),w'(l),\dot w'(l),q,\dot q\big] \tag{1}$$

$$V = V_D(q,\dot q) + \int_{l_0}^{l} \hat V(q,\dot q,\dot w,w,w',w'')\,\mathrm{d}x + V_B\big[w(l),\dot w(l),w'(l),\dot w'(l),q,\dot q\big] \tag{2}$$

so that

$$\begin{aligned} L &= T - V \\ &= L_D(q,\dot q) + \int_{l_0}^{l} \hat L(q,\dot q,\dot w,w,w',w'')\,\mathrm{d}x \\ &\quad + L_B\big[w(l),\dot w(l),w'(l),\dot w'(l),q,\dot q\big] \end{aligned} \tag{3}$$

where $L_D = T_D - V_D$, $\hat L = \hat T - \hat V$, and $L_B = T_B - V_B$.

The nonconservative virtual work of the hybrid system is given by

$$\delta W_{nc} = Q^T\delta q + \int_{l_0}^{l} \hat f^T(x)\delta w\,\mathrm{d}x + f_1^T\delta w(l) + f_2^T\delta w'(l) \tag{4}$$

where f_1 is the nonconservative force vector applied at the boundary, and f_2 is the nonconservative torque vector applied at the boundary.

The extended Hamilton's principle can be stated as

$$\int_{t_1}^{t_2} (\delta L + \delta W_{nc})\,\mathrm{d}t = 0 \qquad \delta q = \delta w = 0 \text{ at } t = t_1, t_2 \tag{5}$$

The variation of L yields

$$\begin{aligned} \int_{t_1}^{t_2} \delta L\,\mathrm{d}t &= \int_{t_1}^{t_2} \left[\frac{\partial L_D}{\partial q}\delta q + \frac{\partial L_D}{\partial \dot q}\delta \dot q \right. \\ &\quad + \int_{l_0}^{l} \left(\frac{\partial \hat L}{\partial q}\delta q + \frac{\partial \hat L}{\partial \dot q}\delta \dot q + \frac{\partial \hat L}{\partial \dot w}\delta \dot w + \frac{\partial \hat L}{\partial w}\delta w + \frac{\partial \hat L}{\partial w'}\delta w' \right. \\ &\quad \left. + \frac{\partial \hat L}{\partial w''}\delta w'' \right)\mathrm{d}x + \frac{\partial L_B}{\partial q}\delta q + \frac{\partial L_B}{\partial \dot q}\delta \dot q + \frac{\partial L_B}{\partial w(l)}\delta w(l) \\ &\quad \left. + \frac{\partial L_B}{\partial \dot w}\delta \dot w(l) + \frac{\partial L_B}{\partial w'(l)}\delta w'(l) + \frac{\partial L_B}{\partial \dot w'(l)}\delta \dot w'(l) \right]\mathrm{d}t \end{aligned} \tag{6}$$

The symbolic integration by parts is tedious (see Ref. 1) but can be carried to completion to obtain the following results. Equation (5) can be written as

$$\begin{aligned} &\int_{t_1}^{t_2} (\delta L + \delta W_{nc})\,\mathrm{d}t \\ &= \int_{t_1}^{t_2} \left[\frac{\partial L_D}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_D}{\partial \dot q}\right) + \int_{l_0}^{l} \frac{\partial \hat L}{\partial q}\,\mathrm{d}x - \int_{l_0}^{l} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \hat L}{\partial \dot q}\right)\mathrm{d}x \right. \\ &\quad \left. + \frac{\partial L_B}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_B}{\partial \dot q}\right) + Q^T \right]\delta q\,\mathrm{d}t + \int_{t_1}^{t_2}\int_{l_0}^{l} \left[-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \hat L}{\partial \dot w}\right) \right. \\ &\quad \left. + \frac{\partial \hat L}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial \hat L}{\partial w'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat L}{\partial w''}\right) + \hat f^T \right]\delta w\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_{t_1}^{t_2} \left(\left[\frac{\partial \hat L}{\partial w'} - \frac{\partial}{\partial x}\left(\frac{\partial \hat L}{\partial w''}\right) \right]\delta w \Big|_{l_0}^{l} \right. \\ &\quad + \left\{ \frac{\partial L_B}{\partial w(l)} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L_B}{\partial \dot w(l)}\right] \right\}\delta w(l) + \left. f_1^T\delta w(l) \right)\mathrm{d}t \\ &\quad + \int_{t_1}^{t_2} \left(\frac{\partial \hat L}{\partial w''}\delta w' \Big|_{l_0}^{l} + \left\{ \frac{\partial L_B}{\partial w'(l)} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L_B}{\partial \dot w'(l)}\right] \right\}\delta w'(l) \right. \\ &\quad \left. + f_2^T\delta w'(l) \right)\mathrm{d}t = 0 \end{aligned} \tag{7}$$

In the preceding equation and elsewhere in the present paper, if F is a function of t, x, $q(t)$, $\dot q(t)$, $w(x,t)$, $\dot w(x,t)$, $w'(x,t)$, $w''(x,t)$, $w(l)$, $\dot w(l)$, $w'(l)$, and $\dot w'(l)$, then the derivative $\mathrm{d}F/\mathrm{d}t$ is defined as

$$\begin{aligned} \frac{\mathrm{d}F}{\mathrm{d}t} &\equiv \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\partial F}{\partial \dot q}\frac{\mathrm{d}\dot q}{\mathrm{d}t} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial t} + \frac{\partial F}{\partial \dot w}\frac{\partial \dot w}{\partial t} \\ &\quad + \frac{\partial F}{\partial w'}\frac{\partial w'}{\partial t} + \frac{\partial F}{\partial w''}\frac{\partial w''}{\partial t} + \frac{\partial F}{\partial w(l)}\frac{\mathrm{d}w(l)}{\mathrm{d}t} \\ &\quad + \frac{\partial F}{\partial \dot w(l)}\frac{\mathrm{d}\dot w(l)}{\mathrm{d}t} + \frac{\partial F}{\partial w'(l)}\frac{\mathrm{d}w'(l)}{\mathrm{d}t} + \frac{\partial F}{\partial \dot w'(l)}\frac{\mathrm{d}\dot w'(l)}{\mathrm{d}t} \end{aligned}$$

Thus, $\mathrm{d}F/\mathrm{d}t$ refers to the partial derivative of F, regarding it as "a function of the independent variables t and x," whereas $\partial F/\partial t$ refers to the so-called "explicit" partial derivative of F regarded as a function of the independent variables t, x, q, $\dot q$, w, $\dot w$, w', w'', $w(l)$, $\dot w(l)$, $w'(l)$, and $\dot w'(l)$. If t does not appear explicitly, then of course $\partial F/\partial t = 0$. If F does not depend on w (and derivatives thereof), then $\mathrm{d}F/\mathrm{d}t$ reduces to the usual definition of total derivative of $F(t,q,\dot q)$. In our original developments we denoted $\mathrm{d}/\mathrm{d}t(\ )$ by $\delta/\delta t(\ )$, but decided this notation was confusing because of the use of δ to denote variations.

Using the usual arguments on the arbitrariness and independence of the variations $\delta q(t)$, $\delta w(x,t)$, and the boundary variations, the preceding equation gives the governing differential equations and the boundary conditions. First, we consider the integrand associated with δq. Since L is expressed by Eq. (3), the first term of Eq. (7) is

$$\int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q}\right) + Q^T \right]\delta q\,\mathrm{d}t$$

Therefore, based on the arbitrariness of δq and the independence of δq, δw, and the boundary variations, we conclude that the preceding integral, hence, the bracketed integrand, must vanish, so that we obtain the classical form for Lagrange's equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot q}\right) - \frac{\partial L}{\partial q} = Q^T \tag{8}$$

We see that the discrete coordinates satisfy the usual form of Lagrange's equations. However, because of the integrations of Eq. (3) over the elastic domain, L and therefore the resulting differential equations must be considered functions of the discrete coordinates, the elastic coordinates, and their space-time derivatives.

To obtain the partial differential equations governing $w(x,t)$, we consider the integrand associated with δw. The second term of Eq. (7) becomes

$$\int_{t_1}^{t_2}\int_{l_0}^{l} \left[-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \hat L}{\partial \dot w}\right) + \frac{\partial \hat L}{\partial w} - \frac{\partial}{\partial x}\left(\frac{\partial \hat L}{\partial w'}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat L}{\partial w''}\right) + \hat f^T \right]\delta w\,\mathrm{d}x\,\mathrm{d}t$$

Because δw is arbitrary and independent of δq and the boundary variations, the bracketed term must vanish, and this provides the partial differential equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \hat{L}}{\partial \dot{w}}\right) - \frac{\partial \hat{L}}{\partial w} + \frac{\partial}{\partial x}\left(\frac{\partial \hat{L}}{\partial w'}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat{L}}{\partial w''}\right) = \hat{f}^T \tag{9}$$

Equations (8) and (9) can be found in Refs. 2 and 7, and these are derived in Ref. 2 for a three-dimensional case.

Next, we consider the boundary conditions. From the last two terms of Eq. (7), we obtain the following symbolic variational statements from which the spatial boundary conditions can be obtained:

$$\left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x}\left(\frac{\partial \hat{L}}{\partial w''}\right)\right]\delta w\Big|_{l_0}^{l} + \left\{\frac{\partial L_B}{\partial w(l)} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L_B}{\partial \dot{w}(l)}\right]\right\}\delta w(l) + f_1^T\delta w(l) = 0 \tag{10}$$

$$\frac{\partial \hat{L}}{\partial w''}\,\delta w'\Big|_{l_0}^{l} + \left\{\frac{\partial L_B}{\partial w'(l)} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L_B}{\partial \dot{w}'(l)}\right]\right\}\delta w'(l) + f_2^T\delta w'(l) = 0 \tag{11}$$

Equations (8) and (9) generate directly the coupled hybrid system of ordinary and partial differential equations, and the variational statements of Eqs. (10) and (11) directly generate the associated boundary conditions. Thus, we have an explicit generalization of Lagrange's equations for nonconservative distributed parameter systems that have one elastic domain but that may undergo large motions and have discrete boundary masses, forces, and moments.

Now we consider a more general case that has more than one elastic domain, i.e., we consider a system of flexible bodies. For simplicity, we consider each elastic body to be beamlike, with only one spatial variable (x_i). Analogous, but more complicated, developments can be carried out in this case. Let us assume that the kinetic energy, potential energy, and Lagrangian are expressed by Eqs. (12), (13), and (14), respectively. We introduce n one-dimensional elastic domains D_i $(i=1,\ldots,n)$, $x_i \in D_i$, where $l_{0_i} < x_i \le l_i$, and $\hat{T}^i$, $\hat{V}^i$, and w_i are defined in domain D_i. For notational compaction, we define $\underline{w}(\underline{l})$ as $w_1(l_1)$, $w_2(l_2),\ldots,w_n(l_n)$; and $\underline{\dot{w}}(\underline{l})$, $\underline{w}'(\underline{l})$, and $\underline{\dot{w}}'(\underline{l})$ are defined in a similar manner. In the general case, $\hat{T}^i$ and $\hat{V}^i$ are functions of q, $\dot{q}$, w_i, $\dot{w}_i$, w_i', w_i'', $\underline{w}(\underline{l})$, $\underline{\dot{w}}(\underline{l})$, $\underline{w}'(\underline{l})$, and $\underline{\dot{w}}'(\underline{l})$, and T_B and V_B are functions of $\underline{w}(\underline{l})$, $\underline{\dot{w}}(\underline{l})$, $\underline{w}'(\underline{l})$, $\underline{\dot{w}}'(\underline{l})$, q, and $\dot{q}$. We assume the following structure for T and V:

$$T = T_D(q,\dot{q}) + \sum_{i=1}^{n}\int_{l_{0_i}}^{l_i} \hat{T}^i\left[q,\dot{q},\dot{w}_i,w_i,w_i',w_i'',\underline{w}(\underline{l}),\underline{\dot{w}}(\underline{l}),\underline{w}'(\underline{l}),\underline{\dot{w}}'(\underline{l})\right]\mathrm{d}x_i + T_B\left[\underline{w}(\underline{l}),\underline{\dot{w}}(\underline{l}),\underline{w}'(\underline{l}),\underline{\dot{w}}'(\underline{l}),q,\dot{q}\right] \tag{12}$$

$$V = V_D(q,\dot{q}) + \sum_{i=1}^{n}\int_{l_{0_i}}^{l_i} \hat{V}^i\left[q,\dot{q},\dot{w}_i,w_i,w_i',w_i'',\underline{w}(\underline{l}),\underline{\dot{w}}(\underline{l}),\underline{w}'(\underline{l}),\underline{\dot{w}}'(\underline{l})\right]\mathrm{d}x_i + V_B\left[\underline{w}(\underline{l}),\underline{\dot{w}}(\underline{l}),\underline{w}'(\underline{l}),\underline{\dot{w}}'(\underline{l}),q,\dot{q}\right] \tag{13}$$

The Lagrangian is, therefore,

$$L = T - V = L_D(q,\dot{q}) + \sum_{i=1}^{n}\int_{l_{0_i}}^{l_i} \hat{L}^i\left[q,\dot{q},\dot{w}_i,w_i,w_i',w_i'',\underline{w}(\underline{l}),\underline{\dot{w}}(\underline{l}),\underline{w}'(\underline{l}),\underline{\dot{w}}'(\underline{l})\right]\mathrm{d}x_i + L_B\left[\underline{w}(\underline{l}),\underline{\dot{w}}(\underline{l}),\underline{w}'(\underline{l}),\underline{\dot{w}}'(\underline{l}),q,\dot{q}\right] \tag{14}$$

where $L_D = T_D - V_D$, $\hat{L}^i = \hat{T}^i - \hat{V}^i$, and $L_B = T_B - V_B$.

The nonconservative virtual work of the hybrid system is given by

$$\delta W_{nc} = Q^T\delta q + \sum_{i=1}^{n}\left[\int_{l_{0_i}}^{l_i}\hat{f}^{iT}(x_i)\delta w_i\,\mathrm{d}x_i + f_1^{iT}\delta w_i(l_i) + f_2^{iT}\delta w_i'(l_i)\right] \tag{15}$$

where f_1^i is the nonconservative force vector applied at the boundary (at $x_i = l_i$) of domain D_i, and f_2^i is the nonconservative torque vector applied at the boundary of domain D_i.

We use the extended Hamilton's principle that can be stated as

$$\int_{t_1}^{t_2}(\delta L + \delta W_{nc})\,\mathrm{d}t = 0 \qquad \delta q = \delta w_i = 0 \quad \text{at } t = t_1, t_2 \tag{16}$$

After carrying out the variation of L and symbolic integration by parts (see Ref. 1), then Eq. (16) can be written in the following compact form:

$$\begin{aligned}\int_{t_1}^{t_2}(\delta L + \delta W_{nc})\,\mathrm{d}t &= \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + Q^T\right]\delta q\,\mathrm{d}t \\ &+ \sum_{i=1}^{n}\int_{t_1}^{t_2}\int_{l_{0_i}}^{l_i}\left[-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \hat{L}^i}{\partial \dot{w}_i}\right) + \frac{\partial \hat{L}^i}{\partial w_i} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}^i}{\partial w_i'}\right) + \frac{\partial^2}{\partial x_i^2}\left(\frac{\partial \hat{L}^i}{\partial w_i''}\right) + \hat{f}^{iT}\right]\delta w_i\,\mathrm{d}x_i\,\mathrm{d}t \\ &+ \sum_{i=1}^{n}\int_{t_1}^{t_2}\int_{l_{0_i}}^{l_i}\left(\left[\frac{\partial \hat{L}^i}{\partial w_i'} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}^i}{\partial w_i''}\right)\right]\delta w_i\Big|_{l_{0_i}}^{l_i} + \left\{\frac{\partial L_B}{\partial w_i(l_i)} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L_B}{\partial \dot{w}_i(l_i)}\right]\right\}\delta w_i(l_i) + f_1^{iT}\delta w_i(l_i)\right)\mathrm{d}x_i\,\mathrm{d}t \\ &+ \sum_{i=1}^{n}\int_{t_1}^{t_2}\int_{l_{0_i}}^{l_i}\left(\frac{\partial \hat{L}^i}{\partial w_i''}\delta w_i'\Big|_{l_{0_i}}^{l_i} + \left\{\frac{\partial L_B}{\partial w_i'(l_i)} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L_B}{\partial \dot{w}_i'(l_i)}\right]\right\}\delta w_i'(l_i) + f_2^{iT}\delta w_i'(l_i)\right)\mathrm{d}x_i\,\mathrm{d}t \\ &= 0\end{aligned} \tag{17}$$

where

$$L_B \equiv L_B + \sum_{i=1}^{n}\int_{l_{0_i}}^{l_i}\hat{L}^i\,\mathrm{d}x_i \tag{18}$$

$$L = L_D + \sum_{i=1}^{n}\int_{l_{0_i}}^{l_i}\hat{L}^i\,\mathrm{d}x_i + L_B \tag{14}$$

Then by using the previous definitions and the usual arguments on the arbitrariness and independence of the variations, we can obtain the following relatively simple equations that are the generalizations of Eqs. (8–11):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q^T \tag{19}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \hat{L}^i}{\partial \dot{w}_i}\right) - \frac{\partial \hat{L}^i}{\partial w_i} + \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}^i}{\partial w_i'}\right) - \frac{\partial^2}{\partial x_i^2}\left(\frac{\partial \hat{L}^i}{\partial w_i''}\right) = \hat{f}^{iT} \tag{20}$$

$$\left[\frac{\partial \hat{L}^i}{\partial w_i'} - \frac{\partial}{\partial x_i}\left(\frac{\partial \hat{L}^i}{\partial w_i''}\right)\right]\delta w_i\Big|_{l_{0_i}}^{l_i} + \left\{\frac{\partial L_B}{\partial w_i(l_i)} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L_B}{\partial \dot{w}_i(l_i)}\right]\right\}\delta w_i(l_i) + f_1^{iT}\delta w_i(l_i) = 0 \tag{21}$$

$$\frac{\partial \hat{L}^i}{\partial w_i''}\delta w_i' \Bigg|_{l_{0_i}}^{l_i} + \left\{\frac{\partial L_B}{\partial w_i'(l_i)} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial L_B}{\partial \dot{w}_i'(l_i)}\right]\right\}\delta w_i'(l_i) + f_2^{iT}\delta w_i'(l_i) = 0 \tag{22}$$

Equations (19) and (20) generate directly the coupled hybrid system of ordinary and partial differential equations, and the variational statements of Eqs. (21) and (22) directly generate the associated boundary conditions. Actually, Eqs. (8–11) are the special case ($n = 1$) of the preceding equations. Thus, we have an explicit generalization of Lagrange's equations, for a large family of hybrid systems that consist of interconnected rigid and elastic bodies. Each elastic body is to be beamlike and can have several dependent distributed variables, but only one independent spatial variable.

In essence, we have done the integrations by parts once and for all for a large class of systems. Thus, the governing equations become quite analogous to the discrete version of Lagrange's equations, i.e., through appropriate derivatives of energy functions. The utility of these equations [Eqs. (19–22)] can be appreciated by considering several examples.

Illustrative Examples

Simplest Class of Problems

When there is only one domain for the elastic motion and there are no boundary dependent terms in the Lagrangian, then the preceding developments are especially simple. The system Lagrangian L is expressed as $L = L_D + \int_{l_0}^{l} \hat{L}\,\mathrm{d}x$, and then Eqs. (8–11) or, more generally, Eqs. (19–22) are simplified to the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = Q^T \tag{8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \hat{L}}{\partial \dot{w}}\right) - \frac{\partial \hat{L}}{\partial w} + \frac{\partial}{\partial x}\left(\frac{\partial \hat{L}}{\partial w'}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial \hat{L}}{\partial w''}\right) = \hat{f}^T \tag{9}$$

$$\left[\frac{\partial \hat{L}}{\partial w'} - \frac{\partial}{\partial x}\left(\frac{\partial \hat{L}}{\partial w''}\right)\right]\delta w \Bigg|_{l_0}^{l} = 0 \tag{23}$$

$$\frac{\partial \hat{L}}{\partial w''}\delta w' \Bigg|_{l_0}^{l} = 0 \tag{24}$$

For this simplest class of problems, it is apparent that the boundary conditions do not make allowance for lumped masses, springs, and similar forces at the boundaries, and obviously the preceding equations do not apply to multilink flexible body chains.

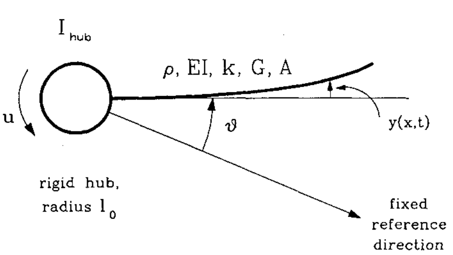

Fig. 1 Rigid hub with a cantilevered Timoshenko beam.

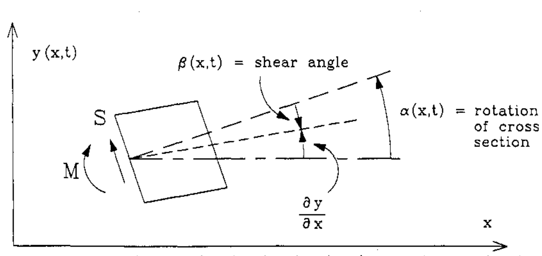

Fig. 2 Kinematics of deformation of a Timoshenko beam.

Rigid Hub with a Cantilevered Timoshenko Beam

With reference to Figs. 1 and 2, we consider a rigid hub with a cantilevered flexible appendage. The appendage is considered to be a uniform flexible beam, and we make the Timoshenko assumptions. The beam is cantilevered rigidly to the hub. Motion is restricted to the horizontal plane, and a control torque $u(t)$ acting on the hub (normal to the plane of motion) is the only external effect. Figure 2 shows the kinematics of deformation of a beam that undergoes shear deformation in addition to pure bending. In this example, we neglect the velocity component $-y\dot{\theta}$, which is perpendicular to the y direction. Under these assumptions, the kinetic and potential energies of this hybrid system are as follows:

$$T = \tfrac{1}{2} I_{\mathrm{hub}}\dot{\theta}^2 + \frac{1}{2}\int_{l_0}^{l}\left[\rho(\dot{y} + x\dot{\theta})^2 + \left(\frac{\rho I}{A}\right)(\dot{\alpha} + \dot{\theta})^2\right]\mathrm{d}x$$

$$V = \frac{1}{2}\int_{l_0}^{l}\left[EI(\alpha')^2 + kGA(\alpha - y')^2\right]\mathrm{d}x$$

where

E = Young's modulus of the beam
I = moment of inertia of cross section about centroidal axis
ρ = constant mass/unit length of the beam
k = shear coefficient
G = modulus of rigidity
A = area of cross section on which shear force acts
I_{hub} = moment of inertia of the rigid hub
θ = hub inertial roration
y = elastic deformation
α = rotation of cross section

Therefore, the Lagrangian is expressed by following equation:

$$\begin{aligned} L &= L_D + \int_{l_0}^{l}\hat{L}\,\mathrm{d}x \\ &= \tfrac{1}{2} I_{\mathrm{hub}}\dot{\theta}^2 + \frac{1}{2}\int_{l_0}^{l}\left[\rho(\dot{y} + x\dot{\theta})^2 + \left(\frac{\rho I}{A}\right)(\dot{\alpha} + \dot{\theta})^2 \right. \\ &\quad \left. - EI(\alpha')^2 - kGA(\alpha - y')^2\right]\mathrm{d}x \end{aligned}$$

The discrete and distributed coordinates for this case are

$$q(t) = \theta, \qquad w(x,t) = [y \quad \alpha]^T$$

and the only external force is $u(t)$, so $Q = u$ and $\hat{f} = 0$.

From Eqs. (8) and (9) and the Lagrangian, we get the governing differential equations for this hybrid system:

$$I_{\mathrm{hub}}\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \int_{l_0}^{l}\left[\rho x\left(\frac{\partial^2 y}{\partial t^2} + x\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right) + \left(\frac{\rho I}{A}\right)\left(\frac{\partial^2\alpha}{\partial t^2} + \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right)\right]\mathrm{d}x = u$$

$$\rho\left(\frac{\partial^2 y}{\partial t^2} + x\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right) + kGA\left(\frac{\partial\alpha}{\partial x} - \frac{\partial^2 y}{\partial x^2}\right) = 0$$

$$\frac{\rho I}{A}\left(\frac{\partial^2\alpha}{\partial t^2} + \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\right) + kGA\left(\alpha - \frac{\partial y}{\partial x}\right) - EI\frac{\partial^2\alpha}{\partial x^2} = 0$$

Boundary conditions are obtained by Eqs. (23) and (24):

$$kGA\left(\alpha - \frac{\partial y}{\partial x}\right)\delta y \Bigg|_{l_0}^{l} = 0, \qquad -EI\left(\frac{\partial\alpha}{\partial x}\right)\delta\alpha \Bigg|_{l_0}^{l} = 0$$

Since $y(l_0) = 0$, $\alpha(l_0) = 0$, and $\delta y(l)$ and $\delta\alpha(l)$ are free, the

boundary conditions are the following:
At $x = l_0$:

$$y = 0 \quad \text{and} \quad \alpha = 0$$

At $x = l$:

$$EI \frac{\partial \alpha}{\partial x}\bigg|_l = 0$$

$$kGA\left(\alpha - \frac{\partial y}{\partial x}\right)\bigg|_l = 0$$

Extensions to Include General Boundary Conditions

We consider the case in which the kinetic, potential, and Lagrangian energy functionals depend on the boundary motion (at $x = l$). So there exist L_B, f_1, and f_2. The boundary conditions presented in this section make allowance for the lumped masses and the springs at the boundaries. We also assume that there is only one domain for the elastic motion. So L is expressed as $L = L_D + \int_{l_0}^{l} \hat{L}\, dx + L_B$. Then we can use Eqs. (8-11).

Flexible Three-Body Problem (Hub, Beam, and Tip Mass)

With reference to Fig. 3, we consider a rigid hub with a cantilevered flexible appendage that has a finite tip mass. The appendage is considered to be a uniform flexible beam, and we make the Euler-Bernoulli assumptions of negligible shear deformation and negligible distributed rotatory inertia. The other assumptions are identical to the first example, except considering the tip mass and the rotatory inertia of the tip mass. The kinetic and potential energies of this hybrid system are as follows:

$$T = ½ I_{\text{hub}}\dot{\theta}^2 + \frac{1}{2}\int_{l_0}^{l}\left[\rho(\dot{y} + x\dot{\theta})^2\right] dx + ½ m\left[l\dot{\theta} + \dot{y}(l)\right]^2 + ½ I_{\text{tip}}\left[\dot{\theta} + \dot{y}'(l)\right]^2$$

$$V = \frac{1}{2}\int_{l_0}^{l}\left[EI(y'')^2\right] dx$$

where m is the mass of the tip mass and I_{tip} the rotatory inertia of the tip mass. Therefore, the Lagrangian may be expressed as

$$L = L_D + \int_{l_0}^{l} \hat{L}\, dx + L_B = ½ I_{\text{hub}}\dot{\theta}^2 + \frac{1}{2}\int_{l_0}^{l}\left[\rho(\dot{y} + x\dot{\theta})^2 - EI(y'')^2\right] dx + ½ m\left[l\dot{\theta} + \dot{y}(l)\right]^2 + ½ I_{\text{tip}}\left[\dot{\theta} + \dot{y}'(l)\right]^2$$

The discrete and distributed coordinates for this case are

$$q(t) = \theta, \qquad w(x,t) = y$$

and the only external force is $u(t)$, so $Q = u$, $\hat{f} = 0$, $f_1 = 0$, and $f_2 = 0$.

From Eqs. (8) and (9), the governing equations for this hybrid system are

$$I_{\text{hub}}\frac{d^2\theta}{dt^2} + \int_{l_0}^{l}\rho x\left(\frac{\partial^2 y}{\partial t^2} + x\frac{d^2\theta}{dt^2}\right) dx + ml\left(l\frac{d^2\theta}{dt^2} + \frac{\partial^2 y}{\partial t^2}\bigg|_l\right) + I_{\text{tip}}\left[\frac{d^2\theta}{dt^2} + \frac{d^2}{dt^2}\left(\frac{\partial y}{\partial x}\bigg|_l\right)\right] = u$$

$$\rho\left(\frac{\partial^2 y}{\partial t^2} + x\frac{d^2\theta}{dt^2}\right) + EI\frac{\partial^4 y}{\partial x^4} = 0$$

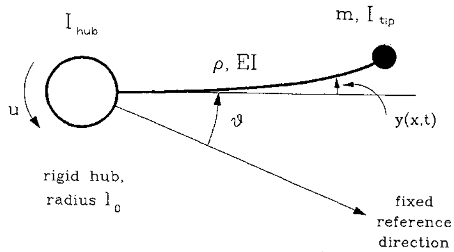

Fig. 3 Flexible three-body problem (hub, beam, and tip mass).

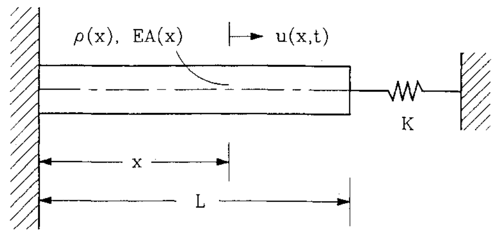

Fig. 4 Rod in axial vibration.

Boundary conditions are obtained from Eqs. (10) and (11) by inserting the descriptive variables,

$$EI\frac{\partial^3 y}{\partial x^3}\delta y\bigg|_{l_0}^{l} - m\left[l\ddot{\theta} + \ddot{y}(l)\right]\delta y(l) = 0$$

$$-EI\frac{\partial^2 y}{\partial x^2}\delta\left(\frac{\partial y}{\partial x}\right)\bigg|_{l_0}^{l} - I_{\text{tip}}\left[\ddot{\theta} + \ddot{y}'(l)\right]\delta y'(l) = 0$$

Since $y(l_0) = y'(l_0) = 0$, and $\delta y(l)$ and $\delta y'(l)$ are free, the boundary conditions are the following:
At $x = l_0$:

$$y = 0 \quad \text{and} \quad y' = 0$$

At $x = l$:

$$EI\frac{\partial^3 y}{\partial x^3}\bigg|_l = m\left[l\ddot{\theta} + \ddot{y}(l)\right] \qquad \text{(shear force)}$$

$$EI\frac{\partial^2 y}{\partial x^2}\bigg|_l = -I_{\text{tip}}\left[\ddot{\theta} + \ddot{y}'(l)\right] \qquad \text{(bending moment)}$$

Axial Vibration of a Rod

With reference to Fig. 4, we consider the rod in axial vibration.[7] For simplicity, we assume that the rod has uniform properties along the axial coordinates. In this case, there are no discrete coordinates. Then the kinetic and potential energies of this system are as follows:

$$T = \frac{1}{2}\int_0^L \rho\dot{u}^2\, dx$$

$$V = \frac{1}{2}\int_0^L EA(u')^2\, dx + ½ K u^2(L)$$

where E is Young's modulus of the rod, A the area of cross section, K the spring constant, ρ the constant mass/unit length of the rod, and u the axial displacement $u = u(x,t)$. Therefore, the Lagrangian may be expressed as

$$L = \int_0^L \hat{L}\, dx + L_B = \frac{1}{2}\int_0^L\left[\rho\dot{u}^2 - EA(u')^2\right] dx - ½ K u^2(L)$$

and there is no external force, so $Q = 0$, $\hat{f} = 0$, $f_1 = 0$, $f_2 = 0$, and $w(x,t) = u(x,t)$. Governing equations for this system follow immediately from the Eq. (9) as

$$\rho \ddot{u} - EA\frac{\partial^2 u}{\partial x^2} = 0$$

and the boundary condition variational statement is obtained by Eq. (10):

$$-EAu'\delta u\Big|_0^L + \left[-Ku(L)\right]\delta u(L) = 0$$

Since $u(0)=0$, and $\delta u(L)$ is free, the boundary conditions are as follows:

At $x=0$:

$$u = 0$$

At $x=L$:

$$EAu'\Big|_L + Ku\Big|_L = 0$$

More General Cases: Multiple-Connected Elastic Bodies

We consider more general cases in which L is expressed as

$$L = L_D + \sum_{i=1}^{n}\int_{l_{0_i}}^{l_i}\hat{\mathcal{L}}^i\,\mathrm{d}x_i + L_B$$

There is more than one elastic domain, i.e., we consider a system of flexible bodies. The governing equations for the hybrid system are obtained from Eqs. (19) and (20), and the boundary conditions are obtained from Eqs. (21) and (22).

Two-Link Flexible Manipulator Model

We consider a planar two-link flexible manipulator as shown in Fig. 5. Each link is modeled as a uniform flexible beam, and we make the Euler-Bernoulli assumptions. This system has two spatial variables x_1 and x_2, and elastic motions relative to the rigid-body motions are described by $y_1(x_1,t)$ and $y_2(x_2,t)$, respectively. In this example, we neglect the velocity components that are perpendicular to the y_1 and y_2 directions. This system is controlled by the torque inputs u_1 and u_2 as shown in Fig. 5.

The kinetic and potential energies of this hybrid system are as follows:

$$T = \frac{1}{2}\int_0^{l_1}\rho_1(x_1\dot{\theta}_1+\dot{y}_1)^2\,\mathrm{d}x_1 + \tfrac{1}{2}m_2\left[l_1\dot{\theta}_1+\dot{y}_1(l_1,t)\right]^2$$
$$+\frac{1}{2}\int_0^{l_2}\rho_2(x_2\dot{\theta}_2+\dot{y}_2)^2\,\mathrm{d}x_2$$
$$+\left[l_1\dot{\theta}_1+\dot{y}_1(l_1,t)\right]\cos(\theta_2-\theta_1)\int_0^{l_2}\rho_2(x_2\dot{\theta}_2+\dot{y}_2)\,\mathrm{d}x_2$$

$$V = \frac{1}{2}\int_0^{l_1}EI_1\left(\frac{\partial^2 y_1}{\partial x_1^2}\right)^2\mathrm{d}x_1 + \frac{1}{2}\int_0^{l_2}EI_2\left(\frac{\partial^2 y_2}{\partial x_2^2}\right)^2\mathrm{d}x_2$$

where ρ_i is the assumed constant mass/unit length of the ith beam, EI_i the assumed constant bending stiffness of the ith beam, l_i the length of the ith beam, y_i the elastic deformation of the ith beam, and m_2 the mass of the second beam.

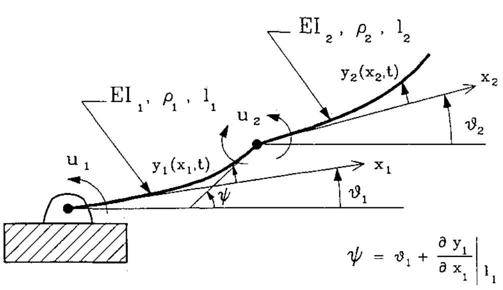

Fig. 5 Two-link flexible manipulator.

$$L = \int_0^{l_1}\hat{\mathcal{L}}^1\,\mathrm{d}x_1 + \int_0^{l_2}\hat{\mathcal{L}}^2\,\mathrm{d}x_2 + L_B$$
$$= \int_0^{l_1}\left[\tfrac{1}{2}\rho_1(x_1\dot{\theta}_1+\dot{y}_1)^2 - \tfrac{1}{2}EI_1\left(\frac{\partial^2 y_1}{\partial x_1^2}\right)^2\right]\mathrm{d}x_1$$
$$+\int_0^{l_2}\left\{\tfrac{1}{2}\rho_2(x_2\dot{\theta}_2+\dot{y}_2)^2 + \left[l_1\dot{\theta}_1+\dot{y}_1(l_1,t)\right]\cos(\theta_2-\theta_1)\right.$$
$$\left.\times\,\rho_2(x_2\dot{\theta}_2+\dot{y}_2) - \tfrac{1}{2}EI_2\left(\frac{\partial^2 y_2}{\partial x_2^2}\right)^2\right\}\mathrm{d}x_2$$
$$+\tfrac{1}{2}m_2\left[l_1\dot{\theta}_1+\dot{y}_1(l_1,t)\right]^2$$

The discrete and distributed coordinates for this case are

$$q_1 = \theta_1(t), \qquad q_2 = \theta_2(t)$$
$$w_1 = y_1(x_1,t), \qquad w_2 = y_2(x_2,t)$$

The nonconservative virtual work is expressed by the following equation:

$$\delta W_{nc} = u_1\delta\theta_1 + u_2\delta\left[\theta_2 - \left(\theta_1 + \frac{\partial y_1}{\partial x_1}\Big|_{l_1}\right)\right]$$
$$= (u_1-u_2)\delta\theta_1 + u_2\delta\theta_2 - u_2\delta\left(\frac{\partial y_1}{\partial x_1}\Big|_{l_1}\right)$$

Therefore,

$$Q = [u_1-u_2 \quad u_2]^T, \qquad \hat{f}^1 = 0, \qquad \hat{f}^2 = 0$$

and

$$f_1^1 = 0, \qquad f_1^2 = 0, \qquad f_2^1 = -u_2, \qquad f_2^2 = 0$$

To apply Eqs. (19–22), first we record $\hat{\mathcal{L}}^1$, $\hat{\mathcal{L}}^2$, and L_B. In this example, L_B is identical to L since L_D does not exist:

$$\hat{\mathcal{L}}^1 = \tfrac{1}{2}\rho_1(x_1\dot{\theta}_1+\dot{y}_1)^2 - \tfrac{1}{2}EI_1\left(\frac{\partial^2 y_1}{\partial x_1^2}\right)^2$$
$$\hat{\mathcal{L}}^2 = \tfrac{1}{2}\rho_2(x_2\dot{\theta}_2+\dot{y}_2)^2 - \tfrac{1}{2}EI_2\left(\frac{\partial^2 y_2}{\partial x_2^2}\right)^2$$
$$+\left[l_1\dot{\theta}_1+\dot{y}_1(l_1,t)\right]\cos(\theta_2-\theta_1)\rho_2(x_2\dot{\theta}_2+\dot{y}_2)$$
$$L_B = \tfrac{1}{2}m_2\left[l_1\dot{\theta}_1+\dot{y}_1(l_1,t)\right]^2 + \sum_{i=1}^{2}\int_0^{l_i}\hat{\mathcal{L}}^i\,\mathrm{d}x_i$$

The resulting coupled system of nonlinear hybrid differential equations and the boundary conditions are obtained from Eqs. (19–22) as the following eight equations:

θ_1 equation:

$$\int_0^{l_1}\rho_1 x_1(x_1\ddot{\theta}_1+\ddot{y}_1)\,\mathrm{d}x_1 + m_2 l_1\left[l_1\ddot{\theta}_1+\ddot{y}_1(l_1,t)\right]$$
$$+\,l_1\cos(\theta_2-\theta_1)\int_0^{l_2}\rho_2(x_2\ddot{\theta}_2+\ddot{y}_2)\,\mathrm{d}x_2$$
$$-\sin(\theta_2-\theta_1)\left[l_1\dot{\theta}_2+\dot{y}_1(l_1,t)\right]\int_0^{l_2}\rho_2(x_2\dot{\theta}_2+\dot{y}_2)\,\mathrm{d}x_2 = u_1 - u_2$$

θ_2 equation:

$$\int_0^{l_2}\rho_2 x_2(x_2\ddot{\theta}_2+\ddot{y}_2)\,\mathrm{d}x_2 + \tfrac{1}{2}m_2 l_2\left[l_1\ddot{\theta}_1+\ddot{y}_1(l_1,t)\right]\cos(\theta_2-\theta_1)$$

$$+\sin(\theta_2-\theta_1)\left[l_1\dot{\theta}_1+\dot{y}_1(l_1,t)\right]\left[\int_0^{l_2}\rho_2(x_2\dot{\theta}_2+\dot{y}_2)\,\mathrm{d}x_2\right.$$

$$\left.-\tfrac{1}{2}m_2l_2(\dot{\theta}_2-\dot{\theta}_1)\right]=u_2$$

y_1 and y_2 equations:

$$\rho_1(x_1\ddot{\theta}_1+\ddot{y}_1)+EI_1\frac{\partial^4 y_1}{\partial x_1^4}=0,\qquad 0<x_1<l_1$$

$$\rho_2(x_2\ddot{\theta}_2+\ddot{y}_2)+EI_2\frac{\partial^4 y_2}{\partial x_2^4}+\rho_2\left[l_1\ddot{\theta}_1+\ddot{y}_1(l_1,t)\right]\cos(\theta_2-\theta_1)$$

$$-\rho_2\left[l_1\dot{\theta}_1+\dot{y}_1(l_1,t)\right](\dot{\theta}_2-\dot{\theta}_1)\sin(\theta_2-\theta_1)=0,\qquad 0<x_2<l_2$$

Boundary conditions of beam 1:

At $x_1=0$:

$$y_1=0,\qquad y_1'=0$$

At $x_1=l_1$:

$$EI_1\frac{\partial^2 y_1}{\partial x_1^2}\bigg|_{l_1}=-u_2$$

$$EI_1\frac{\partial^3 y_1}{\partial x_1^3}\bigg|_{l_1}=m_2\left[l_1\ddot{\theta}_1+\ddot{y}_1(l_1,t)\right]$$

$$+\cos(\theta_2-\theta_1)\int_0^{l_2}\rho_2(x_2\ddot{\theta}_2+\ddot{y}_2)\,\mathrm{d}x_2$$

$$-\sin(\theta_2-\theta_1)(\dot{\theta}_2-\dot{\theta}_1)\int_0^{l_2}\rho_2(x_2\dot{\theta}_2+\dot{y}_2)\,\mathrm{d}x_2$$

Boundary conditions of beam 2:

At $x_2=0$:

$$y_2=0,\qquad y_2'=0$$

At $x_2=l_2$:

$$EI_2\frac{\partial^2 y_2}{\partial x_2^2}\bigg|_{l_2}=0\qquad\text{(bending moment)}$$

$$EI_2\frac{\partial^3 y_2}{\partial x_2^3}\bigg|_{l_2}=0\qquad\text{(shear force)}$$

The algebra and calculus associated with carrying through general developments leading to Eqs. (19–22) is comparable to doing one special case application of Hamilton's principle (for example, the preceding flexible manipulator). However, having derived Eqs. (19–22), we can very efficiently obtain the system equations and boundary conditions for any system that belongs to the family to which Eqs. (19–22) apply.

Large Flexible Space Structure Model

We consider a space structure model that consists of four elastic domains, two boundary elements, and one discrete element, a rigid hub, as presented in Figs. 6 and 7. This model is specifically motivated by the hardware experimental configuration under study at the Naval Postgraduate School by Agrawal et al.[8] Each elastic domain consists of a beam element connected through two discrete mass elements. The only rigid-body motion is rotation θ of the hub, and this rigid-body motion forms highly nonlinear coupling effects with other substructures. In this example, we include the velocity components in the x direction as well, which result in a centrifugal force effect. Three control torque actuators are assumed to be available as in Fig. 8: one (u_1) at the rigid hub and the other two (u_2, u_3) at the discrete mass elements. Recently, Junkins and Bang[9] have used this model for application of their globally stabilizing control law. The kinetic and potential energies for this system are expressed as functions of the position and velocity coordinates as

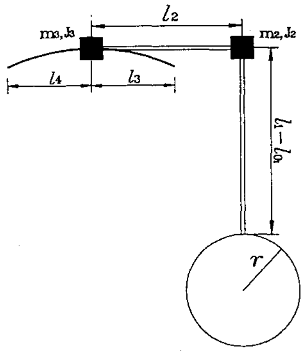

Fig. 6 Undeformed configuration of the structure.

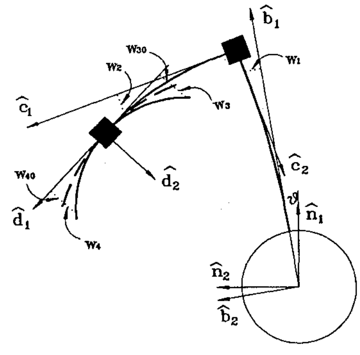
Fig. 7 Deformed configuration of the structure.

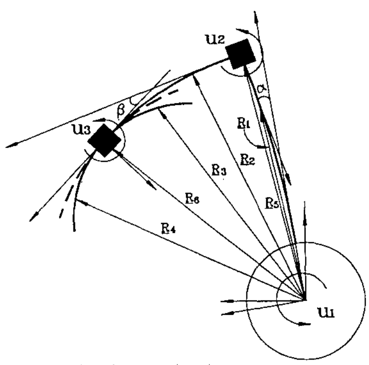

Fig. 8 Position vectors and applied torques.

$$2T=\sum_{i=1}^{2}\int_{l_{0_i}}^{l_i}\left[(\rho_i\dot{R}_i)(\dot{R}_i\,\mathrm{d}x_i)+I_h\dot{\theta}^2+m_2\dot{R}_5(\dot{R}_5)\right.$$

$$\left.+m_3\dot{R}_6(\dot{R}_6)\right]$$

and

$$2V=\sum_{i=1}^{2}\int_{l_{0_i}}^{l_i}E_iI_i\left(\frac{\partial^2 w_i}{\partial x_i^2}\right)^2\mathrm{d}x_i+\sum_{i=3}^{4}\int_{l_{0_i}}^{l_i}E_iI_i\left(\frac{\partial^2 \bar{w}_i}{\partial x_i^2}\right)^2\mathrm{d}x_i$$

The position vectors for the four flexible appendages and two discrete mass elements can be represented as in Fig. 8 with respect to the body-fixed axes $\hat{b}_1$ and $\hat{b}_2$ as follows:

$$\underline{R}_1=x_1\hat{b}_1+w_1\hat{b}_2$$

$$\underline{R}_2=(l_1-x_2\sin\alpha-w_2\cos\alpha)\hat{b}_1$$

$$+\left[w_1(l_1)+x_2\cos\alpha-w_2\sin\alpha\right]\hat{b}_2$$

$$\underline{R}_3 = \left[l_1 - l_2\sin\alpha - w_2(l_2)\cos\alpha - x_3\sin(\alpha+\beta) - w_3\cos(\alpha+\beta)\right]\hat{b}_1 + \left[w_1(l_1) + l_2\cos\alpha - w_2(l_2)\sin\alpha - x_3\cos(\alpha+\beta) - w_3\sin(\alpha+\beta)\right]\hat{b}_2$$

$$\underline{R}_4 = \left[l_1 - l_2\sin\alpha - w_2(l_2)\cos\alpha - x_4\sin(\alpha+\beta) - w_4\cos(\alpha+\beta)\right]\hat{b}_1 + \left[w_1(l_1) + l_2\cos\alpha - w_2(l_2)\sin\alpha + x_4\cos(\alpha+\beta) - w_4\sin(\alpha+\beta)\right]\hat{b}_2$$

$$\underline{R}_5 = l_1\hat{b}_1 + w_1(l_1)\hat{b}_2$$

$$\underline{R}_6 = \left[l_1 - l_2\sin\alpha - w_2(l_2)\cos\alpha\right]\hat{b}_1 + \left[w_1(l_1) + l_2\cos\alpha - w_2(l_2)\sin\alpha\right]\hat{b}_2$$

where α and β denote slopes of deflections at the tip of first and second elastic domains,

$$\alpha = \left.\frac{\partial w_1}{\partial x_1}\right|_{l_1}, \qquad \beta = \left.\frac{\partial w_2}{\partial x_2}\right|_{l_2}$$

With respect to the inertial frame denoted by N of the position vectors, the velocity vectors are given by

$$\dot{\underline{R}}_1 = \frac{{}^N\mathrm{d}\underline{R}_1}{\mathrm{d}t} = (-w_1\dot\theta)\hat{b}_1 + (\dot w_1 + x_1\dot\theta)\hat{b}_2$$

$$\dot{\underline{R}}_2 = \frac{{}^N\mathrm{d}\underline{R}_2}{\mathrm{d}t} = \Big\{-x_2\dot\alpha\cos\alpha - \dot w_2\cos\alpha + w_2\dot\alpha\sin\alpha - \dot\theta\left[w_1(l_1) + x_2\cos\alpha - w_2\sin\alpha\right]\Big\}\hat{b}_1 + \left[\dot w_1(l_1) - x_2\dot\alpha\sin\alpha - \dot w_2\sin\alpha - w_2\dot\alpha\cos\alpha + \dot\theta(l_1 - x_2\sin\alpha - w_2\cos\alpha)\right]\hat{b}_2$$

$$\dot{\underline{R}}_3 = \frac{{}^N\mathrm{d}\underline{R}_3}{\mathrm{d}t} = \Big\{-l_2\dot\alpha\cos\alpha - \dot w_2(l_2)\cos\alpha + w_2(l_2)\dot\alpha\sin\alpha + x_3(\dot\alpha+\dot\beta)\cos(\alpha+\beta) - \dot w_3\cos(\alpha+\beta) + w_3(\dot\alpha+\dot\beta)\sin(\alpha+\beta) - \dot\theta\left[w_1(l_1) + l_2\cos\alpha - w_2(l_2)\sin\alpha - x_3\cos(\alpha+\beta) - w_3\sin(\alpha+\beta)\right]\Big\}\hat{b}_1 + \Big\{\dot w_1(l_1) - l_2\dot\alpha\sin\alpha - \dot w_2(l_2)\sin\alpha - w_2(l_2)\dot\alpha\cos\alpha + x_3(\dot\alpha+\dot\beta)\sin(\alpha+\beta) - \dot w_3\sin(\alpha+\beta) - w_3(\dot\alpha+\dot\beta)\cos(\alpha+\beta) + \dot\theta\left[l_1 - l_2\sin\alpha - w_2(l_2)\cos\alpha + x_3\sin(\alpha+\beta) - w_3\cos(\alpha+\beta)\right]\Big\}\hat{b}_2$$

$$\dot{\underline{R}}_4 = \frac{{}^N\mathrm{d}\underline{R}_4}{\mathrm{d}t} = \Big\{-l_2\dot\alpha\cos\alpha - \dot w_2(l_2)\cos\alpha + w_2(l_2)\dot\alpha\sin\alpha - x_4(\dot\alpha+\dot\beta)\cos(\alpha+\beta) - \dot w_4\cos(\alpha+\beta) + w_4(\dot\alpha+\dot\beta)\sin(\alpha+\beta) - \dot\theta\left[w_1(l_1) + l_2\cos\alpha - w_2(l_2)\sin\alpha + x_4\cos(\alpha+\beta) - w_4\sin(\alpha+\beta)\right]\Big\}\hat{b}_1 + \Big\{\dot w_1(l_1) - l_2\dot\alpha\sin\alpha - \dot w_2(l_2)\sin\alpha - w_2(l_2)\dot\alpha\cos\alpha - x_4(\dot\alpha+\dot\beta)\sin(\alpha+\beta) - \dot w_4\sin(\alpha+\beta) - w_4(\dot\alpha+\dot\beta)\cos(\alpha+\beta) + \dot\theta\left[l_1 - l_2\sin\alpha - w_2(l_2)\cos\alpha - x_4\sin(\alpha+\beta) - w_4\cos(\alpha+\beta)\right]\Big\}\hat{b}_2$$

$$\dot{\underline{R}}_5 = \frac{{}^N\mathrm{d}\underline{R}_5}{\mathrm{d}t} = -w_1(l_1)\dot\theta\hat{b}_1 + \left[\dot w_1(l_1) + l_1\dot\theta\right]\hat{b}_2$$

$$\dot{\underline{R}}_6 = \frac{{}^N\mathrm{d}\underline{R}_6}{\mathrm{d}t} = \Big\{-l_2\dot\alpha\cos\alpha - \dot w_2(l_2)\cos\alpha + w_2(l_2)\dot\alpha\sin\alpha - \dot\theta\left[w_1(l_1) + l_2\cos\alpha - w_2(l_2)\sin\alpha\right]\Big\}\hat{b}_1 + \Big\{\dot w_1(l_1) - l_2\dot\alpha\sin\alpha - \dot w_2(l_2)\sin\alpha - w_2(l_2)\dot\alpha\cos\alpha + \dot\theta\left[l_1 - l_2\sin\alpha - w_2(l_2)\cos\alpha\right]\Big\}\hat{b}_2$$

For convenience of notation, we introduce

$$\dot{\underline{R}}_i = \frac{{}^N\mathrm{d}\underline{R}_i}{\mathrm{d}t} = v_i^1\hat{b}_1 + v_i^2\hat{b}_2$$

The nonconservative forces are given through the expression of virtual work as follows:

$$\delta W = u_1\delta\theta + u_2\delta(\theta+\alpha) + u_3\delta(\alpha+\beta+\theta) = (u_1+u_2+u_3)\delta\theta + (u_2+u_3)\delta\alpha + u_3\delta\beta$$

Applying Eqs. (19–22) yields the following nonlinear equations of motion.

Nonlinear Equations of Motion

θ equation:

$$I_h\ddot\theta + \int_{l_{0_1}}^{l_1}\rho_1\frac{\mathrm{d}}{\mathrm{d}t}(v_1^1A_1^1 + v_1^2B_1^1)\,\mathrm{d}x_1 + \int_{l_{0_2}}^{l_2}\rho_2\frac{\mathrm{d}}{\mathrm{d}t}(v_2^1A_2^2 + v_2^2B_2^2)\,\mathrm{d}x_2 + \int_{l_{0_3}}^{l_3}\rho_3\frac{\mathrm{d}}{\mathrm{d}t}(v_3^1A_3^2 + v_3^2B_3^2)\,\mathrm{d}x_3 + \int_{l_{0_4}}^{l_4}\rho_4\frac{\mathrm{d}}{\mathrm{d}t}(v_4^1A_4^2 + v_4^2B_4^2)\,\mathrm{d}x_4 + m_2\frac{\mathrm{d}}{\mathrm{d}t}(v_5^1A_5^1 + v_5^2B_5^1) + m_3\frac{\mathrm{d}}{\mathrm{d}t}(v_6^1A_6^2 + v_6^2B_6^2) = u_1 + u_2 + u_3$$

w_1 equation:

$$\rho_1\left[-v_1^1A_1^2 + \frac{\mathrm{d}}{\mathrm{d}t}(v_1^2B_1^2)\right] + E_1I_1\frac{\partial^4 w_1}{\partial x_1^4} = 0$$

$w_1(l_1)$ equation:

$$\int_{l_{0_2}}^{l_2}\rho_2\left[\frac{\mathrm{d}}{\mathrm{d}t}(v_2^2B_2^1) - v_2^1A_2^1\right]\mathrm{d}x_2 + \int_{l_{0_3}}^{l_3}\rho_3\left[\frac{\mathrm{d}}{\mathrm{d}t}(v_3^2B_3^1) - v_3^1A_3^1\right]\mathrm{d}x_3 + \int_{l_{0_4}}^{l_4}\rho_4\left[\frac{\mathrm{d}}{\mathrm{d}t}(v_4^2B_4^1) - v_4^1A_4^1\right]\mathrm{d}x_4 + m_2\left[\frac{\mathrm{d}}{\mathrm{d}t}(v_5^2B_5^2) - v_5^1A_5^2\right] + m_3\left[\frac{\mathrm{d}}{\mathrm{d}t}(v_6^2B_6^1) - v_6^1A_6^1\right] = E_1I_1\left.\frac{\partial^3 w_1}{\partial x_1^3}\right|_{l_1}$$

$\alpha \equiv (\partial w_1/\partial x_1)|_{l_1}$ equation:

$$\int_{l_{0_2}}^{l_2}\rho_2\left[\frac{\mathrm{d}}{\mathrm{d}t}(v_2^1A_2^3 + v_2^2B_2^3) - v_2^1A_2^4 - v_2^2B_2^4\right]\mathrm{d}x_2$$

$$+\int_{l_{03}}^{l_3}\rho_3\left[\frac{d}{dt}(v_3^1A_3^3+v_3^2B_3^3)-v_3^1A_3^4-v_3^2B_3^4\right]dx_3$$
$$+\int_{l_{04}}^{l_4}\rho_4\left[\frac{d}{dt}(v_4^1A_4^3+v_4^2B_4^3)-v_4^1A_4^4-v_4^2B_4^4\right]dx_4$$
$$+m_3\left[\frac{d}{dt}(v_6^1A_6^3)+v_6^2B_6^3)-v_6^1A_6^4-v_6^2B_6^4\right]$$
$$=-E_1I_1\frac{\partial^2 w_1}{\partial x_1^2}\bigg|_{l_1}+u_2+u_3$$

w_2 equation:

$$\rho_2\left[\frac{d}{dt}(v_2^1A_2^5+v_2^2B_2^5)-v_2^1A_2^6-v_2^2B_2^6\right]+E_2I_2\frac{\partial^4 w_2}{\partial x_2^4}=0$$

$w_2(l_2)$ equation:

$$\int_{l_{03}}^{l_3}\rho_3\left[\frac{d}{dt}(v_3^1A_3^5+v_3^2B_3^5)-v_3^1A_3^6-v_3^2B_3^6\right]dx_3$$
$$+\int_{l_{04}}^{l_4}\rho_4\left[\frac{d}{dt}(v_4^1A_4^5+v_4^2B_4^5)-v_4^1A_4^6-v_4^2B_4^6\right]dx_4$$
$$+m_3\left[\frac{d}{dt}(v_6^1A_6^5+v_6^2B_6^5)-v_6^1A_6^6-v_6^2B_6^6\right]$$
$$=E_2I_2\frac{\partial^3 w_2}{\partial x_2^3}\bigg|_{l_2}$$

$\beta\equiv(\partial w_2/\partial x_2)|_{l_2}$ equation:

$$\int_{l_{03}}^{l_3}\rho_3\left[\frac{d}{dt}(v_3^1A_3^7+v_3^2B_3^7)-v_3^1A_3^8-v_3^2B_3^8\right]dx_3$$
$$+\int_{l_{04}}^{l_4}\rho_4\left[\frac{d}{dt}(v_4^1A_4^7+v_4^2B_4^7)-v_4^1A_4^8-v_4^2B_4^8\right]dx_4$$
$$=-E_2I_2\frac{\partial^2 w_2}{\partial x_2^2}\bigg|_{l_2}+u_3$$

w_3 equation:

$$\rho_3\left[\frac{d}{dt}(v_3^1A_3^9+v_3^2B_3^9)-v_3^1A_3^{10}-v_3^2B_3^{10}\right]+E_3I_3\frac{\partial^4\tilde{w}_3}{\partial x_3^4}=0$$

w_4 equation:

$$\rho_4\left[\frac{d}{dt}(v_4^1A_4^9+v_4^2B_4^9)-v_4^1A_4^{10}-v_4^2B_4^{10}\right]+E_4I_4\frac{\partial^4\tilde{w}_4}{\partial x_4^4}=0$$

where $\tilde{w}_3=w_3-w_{30}$ and $\tilde{w}_4=w_4-w_{40}$. The constants A_i^j and B_i^j that are functions of the states related to the overall structure are presented in the Appendix.

Summary and Conclusion

In this paper, emphasis has been placed on the multibody case. An explicit version of the classical Lagrange's equations that cover a large family of multibody hybrid discrete distributed parameter systems is symbolically derived. The resulting equations can be efficiently specialized to obtain not only the hybrid governing integro-differential equations but also the associated boundary conditions. These resulting equations enable us to avoid the very tedious system-specific variational arguments and integration by parts. These equations can be generalized further to consider three-dimensional elastic solid bodies.

Appendix

$$A_1^1=-w_1,\quad A_1^2=-\dot\theta,\quad B_1^1=x_1,\quad B_1^2=1$$
$$A_2^1=-\dot\theta,\quad A_2^2=-w_1(l_1)-x_2\cos\alpha+w_2\sin\alpha$$
$$A_2^3=-x_2\cos\alpha+w_2\sin\alpha$$
$$A_2^4=x_2\dot\alpha\sin\alpha+\dot w_2\sin\alpha+w_2\dot\alpha\cos\alpha+x_2\dot\theta\sin\alpha+w_2\dot\theta\cos\alpha$$
$$A_2^5=-\cos\alpha,\quad A_2^6=(\dot\alpha+\dot\theta)\sin\alpha$$
$$B_2^1=1,\quad B_2^2=l_1-x_2\sin\alpha-w_2\cos\alpha$$
$$B_2^3=-x_2\sin\alpha-w_2\cos\alpha$$
$$B_2^4=-x_2\dot\alpha\cos\alpha-\dot w_2\cos\alpha+w_2\dot\alpha\sin\alpha-x_2\dot\theta\cos\alpha$$
$$+w_2\dot\theta\sin\alpha$$
$$B_2^5=-\sin\alpha,\quad B_2^6=-(\dot\alpha+\dot\theta)\cos\alpha$$
$$A_3^1=-\dot\theta$$
$$A_3^2=-w_1(l_1)-l_2\cos\alpha+w_2(l_2)\sin\alpha-x_3\cos(\alpha+\beta)$$
$$+w_3\sin(\alpha+\beta)$$
$$A_3^3=-l_2\cos\alpha+w_2(l_2)\sin\alpha+x_3\cos(\alpha+\beta)+w_3\sin(\alpha+\beta)$$
$$A_3^4=l_2\dot\alpha\sin\alpha+\dot w_2(l_2)\sin\alpha+w_2(l_2)\dot\alpha\cos\alpha+l_2\dot\theta\sin\alpha$$
$$+w_2(l_2)\dot\theta\cos\alpha-x_3(\dot\alpha+\dot\beta)\sin(\alpha+\beta)+\dot w_3\sin(\alpha+\beta)$$
$$+w_3(\dot\alpha+\dot\beta)\cos(\alpha+\beta)-x_3\dot\theta\sin(\alpha+\beta)+w_3\dot\theta\cos(\alpha+\beta)$$
$$A_3^5=-\cos\alpha,\quad A_3^6=(\dot\alpha+\dot\theta)\sin\alpha$$
$$A_3^7=x_3\cos(\alpha+\beta)+w_3\sin(\alpha+\beta)$$
$$A_3^8=-x_3(\dot\alpha+\dot\beta)\sin(\alpha+\beta)+\dot w_3\sin(\alpha+\beta)$$
$$+w_3(\dot\alpha+\dot\beta)\cos(\alpha+\beta)-x_3\dot\theta\sin(\alpha+\beta)+w_3\dot\theta\cos(\alpha+\beta)$$
$$A_3^9=-\cos(\alpha+\beta),\quad A_3^{10}=(\dot\alpha+\dot\beta+\dot\theta)\sin(\alpha+\beta)$$
$$B_3^1=1$$
$$B_3^2=l_1-l_2\sin\alpha-w_2(l_2)\cos\alpha+x_3\sin(\alpha+\beta)-w_3\cos(\alpha+\beta)$$
$$B_3^3=-l_2\sin\alpha-w_2(l_2)\cos\alpha+x_3\sin(\alpha+\beta)-w_3\cos(\alpha+\beta)$$
$$B_3^4=-l_2\dot\alpha\cos\alpha-\dot w_2(l_2)\cos\alpha+w_2(l_2)\dot\alpha\sin\alpha-l_2\dot\theta\cos\alpha$$
$$+w_2(l_2)\dot\theta\sin\alpha+x_3(\dot\alpha+\dot\beta)\cos(\alpha+\beta)-\dot w_3\cos(\alpha+\beta)$$
$$+w_3(\dot\alpha+\dot\beta)\sin(\alpha+\beta)+x_3\dot\theta\cos(\alpha+\beta)+w_3\dot\theta\sin(\alpha+\beta)$$
$$B_3^5=-\sin\alpha,\quad B_3^6=-(\dot\alpha+\dot\theta)\cos\alpha$$
$$B_3^7=x_3\sin(\alpha+\beta)-w_3\cos(\alpha+\beta)$$
$$B_3^8=x_3(\dot\alpha+\dot\beta)\cos(\alpha+\beta)-\dot w_3\cos(\alpha+\beta)$$
$$+w_3(\dot\alpha+\dot\beta)\sin(\alpha+\beta)+x_3\dot\theta\cos(\alpha+\beta)+w_3\dot\theta\sin(\alpha+\beta)$$
$$B_3^9=-\sin(\alpha+\beta),\quad B_3^{10}=-(\dot\alpha+\dot\beta+\dot\theta)\cos(\alpha+\beta)$$
$$A_4^1=-\dot\theta$$
$$A_4^2=-w_1(l_1)-l_2\cos\alpha+w_2(l_2)\sin\alpha-x_4\cos(\alpha+\beta)$$
$$+w_4\sin(\alpha+\beta)$$

$$A_4^3 = -l_2\cos\alpha + w_2(l_2)\sin\alpha - x_4\cos(\alpha+\beta) + w_4\sin(\alpha+\beta)$$

$$A_4^4 = l_2\dot\alpha\sin\alpha + \dot w_2(l_2)\sin\alpha + w_2(l_2)\dot\alpha\cos\alpha + l_2\dot\theta\sin\alpha$$

$$+ w_2(l_2)\dot\theta\cos\alpha + x_4(\dot\alpha+\dot\beta)\sin(\alpha+\beta) + \dot w_4\sin(\alpha+\beta)$$

$$+ w_4(\dot\alpha+\dot\beta)\cos(\alpha+\beta) + x_4\dot\theta\sin(\alpha+\beta) + w_4\dot\theta\cos(\alpha+\beta)$$

$$A_4^5 = -\cos\alpha, \qquad A_4^6 = (\dot\alpha+\dot\theta)\sin\alpha$$

$$A_4^7 = -x_4\cos(\alpha+\beta) + w_4\sin(\alpha+\beta)$$

$$A_4^8 = x_4(\dot\alpha+\dot\beta)\sin(\alpha+\beta) + \dot w_4\sin(\alpha+\beta)$$

$$+ w_4(\dot\alpha+\dot\beta)\cos(\alpha+\beta) + x_4\dot\theta\sin(\alpha+\beta) + w_4\dot\theta\cos(\alpha+\beta)$$

$$A_4^9 = -\cos(\alpha+\beta), \qquad A_4^{10} = (\dot\alpha+\dot\beta+\dot\theta)\sin(\alpha+\beta)$$

$$B_4^1 = 1$$

$$B_4^2 = l_1 - l_2\sin\alpha - w_2(l_2)\cos\alpha - x_4\sin(\alpha+\beta) - w_4\cos(\alpha+\beta)$$

$$B_4^3 = -l_2\sin\alpha - w_2(l_2)\cos\alpha - x_4\sin(\alpha+\beta) - w_4\cos(\alpha+\beta)$$

$$B_4^4 = -l_2\dot\alpha\cos\alpha - \dot w_2(l_2)\cos\alpha + w_2(l_2)\dot\alpha\sin\alpha - l_2\dot\theta\cos\alpha$$

$$+ w_2(l_2)\dot\theta\sin\alpha - x_4(\dot\alpha+\dot\beta)\cos(\alpha+\beta) - \dot w_4\cos(\alpha+\beta)$$

$$+ w_4(\dot\alpha+\dot\beta)\sin(\alpha+\beta) - x_4\dot\theta\cos(\alpha+\beta) + w_4\dot\theta\sin(\alpha+\beta)$$

$$B_4^5 = -\sin\alpha, \qquad B_4^6 = -(\dot\alpha+\dot\theta)\cos\alpha$$

$$B_4^7 = -x_4\sin(\alpha+\beta) - w_4\cos(\alpha+\beta)$$

$$B_4^8 = -x_4(\dot\alpha+\dot\beta)\cos(\alpha+\beta) - \dot w_4\cos(\alpha+\beta)$$

$$+ w_4(\dot\alpha+\dot\beta)\sin(\alpha+\beta) - x_4\dot\theta\cos(\alpha+\beta) + w_4\dot\theta\sin(\alpha+\beta)$$

$$B_4^9 = -\sin(\alpha+\beta), \qquad B_4^{10} = -(\dot\alpha+\dot\beta+\dot\theta)\cos(\alpha+\beta)$$

$$A_5^1 = -w_1(l_1), \qquad A_5^2 = -\dot\theta, \qquad B_5^1 = l_1, \qquad B_5^2 = 1$$

$$A_6^1 = -\dot\theta, \qquad A_6^2 = -w_1(l_1) - l_2\cos\alpha + w_2(l_2)\sin\alpha$$

$$A_6^3 = -l_2\cos\alpha + w_2(l_2)\sin\alpha$$

$$A_6^4 = l_2\dot\alpha\sin\alpha + \dot w_2(l_2)\sin\alpha + w_2(l_2)\dot\alpha\cos\alpha$$

$$+ l_2\dot\theta\sin\alpha + w_2(l_2)\dot\theta\cos\alpha$$

$$A_6^5 = -\cos\alpha, \qquad A_6^6 = (\dot\alpha+\dot\theta)\sin\alpha$$

$$B_6^1 = 1, \qquad B_6^2 = l_1 - l_2\sin\alpha - w_2(l_2)\cos\alpha$$

$$B_6^3 = -l_2\sin\alpha - w_2(l_2)\cos\alpha$$

$$B_6^4 = -l_2\dot\alpha\cos\alpha - \dot w_2(l_2)\cos\alpha + w_2(l_2)\dot\alpha\sin\alpha - l_2\dot\theta\cos\alpha$$

$$+ w_2(l_2)\dot\theta\sin\alpha$$

$$B_6^5 = -\sin\alpha, \qquad B_6^6 = -(\dot\theta+\dot\alpha)\cos\alpha$$

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